이화여자대학교 대학원
2 0 1 8년도
석사학위 청구논문

Solving Nonconvex Quadratic Constrained Quadratic Problems with Hollow Matrices

수 학 과
이 목 화

2019

Solving Nonconvex Quadratic Constrained Quadratic Problems with Hollow Matrices

이 논문을 석사학위 논문으로 제출함

2019 년 6 월

이화여자대학교 대학원

수 학 과 이 목 화

이목화의 석사학위 논문을 인준함

지도교수	김 선 영	
심사위원	윤 정 호	
	김 미 정	
	김 선 영	

이화여자대학교 대학원

Contents

1 Introduction **1**

2 Preliminaries **4**

2.1 Notation and general QCQP 4

2.2 SDP relaxation 5

2.3 SOCP relaxation 6

2.4 LP relaxation 8

3 The equivalence among the SDP, SOCP, and LP relaxations . . . **9**

3.1 The pooling problem 9

3.2 The equivalence of the optimal values of the three relaxations . . . 10

4 Proposed method **15**

4.1 Rearrangement of diagonal elements 15

4.2 Analysis of the proposed method 23

5 Numerical Results **27**

6 Conclusion **33**

References **34**

Abstract

Finding the unconstrained global minimum of given quadratically constrained quadratic programs (QCQPs) is a crucial issue in many class of applications such as binary quadratic optimization problem, the max-cut problems and network problems. There have been a variety of tools and structures to find an optimal solution from various angles. Nevertheless, no one has presented general algorithms to solve the NP-hard problem in polynomial time because of its complexity. Accordingly, rather than finding the exact optimal values, general relaxation approaches have been proposed and their computational and theoretical effectiveness was demonstrated with numerical results by many researchers. Most notable relaxation methods are semi-definite programming (SDP), second order cone programming (SOCP), and linear programming (LP) relaxations. We propose a computational method to solve QCQPs for improved computational efficiency using matrix sparsity. In particular, we employ the particular relationship between SDP, SOCP and LP relaxations to apply the equivalence of the optimal values under a certain assumption used in solving the pooling problem by Kimizuka, Kim and Yamashita, 2018.

1 Introduction

We consider quadratically constrained quadratic Programs (QCQPs) in the form of

$$
\begin{aligned}
\text{(QCQP)} \min_{x} \quad & \boldsymbol{x}^T\boldsymbol{Q}_0\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{q}_0 + r_0 \\
\text{subject to} \quad & \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{q}_i + r_i \leq 0 \quad (i = 1, ..., k), \\
& \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},
\end{aligned} \tag{1}
$$

where $\boldsymbol{x}^T$ denotes the vector transpose of $\boldsymbol{x}$, $\boldsymbol{Q}$ and $\boldsymbol{Q}_i's$ are symmetric matrices in $\mathbb{R}^{n\times n}$, $\boldsymbol{A}$ is a matrix in $\mathbb{R}^{m\times n}$, $\boldsymbol{x}$, $\boldsymbol{q}$, and $\boldsymbol{q}_i's$ are vectors in $\mathbb{R}^n$, $\boldsymbol{b}$ is a vector in $\mathbb{R}^m$, $\boldsymbol{r}$ and $\boldsymbol{r}_i's$ are scalars in $\mathbb{R}$, and k is a natural number.

It is known that if all $\boldsymbol{Q}_i$ $(i = 0, \ldots, k)$ are positive semidefinite, then the problem (1) is a convex problem. Convex problems can be solved exactly with many existing software packages, for instance, SeDuMi, SDPT3, MOSEK [9, 10, 3]. On the other hand, if any of $\boldsymbol{Q}_i$ $(i = 0, \ldots, k)$ are negative semidefinite, then the problem becomes nonconvex QCQP. Nonconvex QCQPs are known to be NP-hard. Thus, providing an accurate approximate solution to nonconvex QCQPs is a very important issue in the field optimization.

Nonconvex QCQPs appear widely in the fields such as graph theory, control theory, robotics, and network problems. More specifically, well-known Max-cut problems can be cast in the form of QCQP (1). Maximum stable set problems [20], maximum-clique problems [7] are also much studied QCQP problems. In addition, quadratic assignment problems are known to be one of the most difficult problems and a great deal of research have conducted to address the computational issues of the problem [5].

One of the fundamental approaches to solve the problems mentioned above is convex relaxation method. Popular convex relaxation methods include semidefinite programming (SDP), second order cone programming (SOCP), and linear programming (LP). These methods can be used to solve QCQP (1) by utilizing mathematical packages such as SeDuMi [9].

SDP relaxations are very accurate in that they provide tighter bounds for the optimal value. Goemans and Williamson [13] showed that using SDP relaxations can yield an approximated result of the Max-cut problem, and Lasserre [8] proposed a hierarchy of semidefinite programming relaxations. In 2012, Kim and Kojima exploited the sparsity of polynomial optimization problems by SDP relaxation [16]. Although the primal-dual interior-point method in SDP relaxation [14] is powerful for searching global solution, it is computationally more expensive as the problem size is getting larger.

Therefore, lift-and-project based SOCP and LP relaxations could be better ways to overcome disadvantages in solving large-sized SDPs. In [17], Kim and Kojima proved that SOCP relaxation is an efficient method for finding good optimal values of QOPs within a reasonable amount of computational time. Muramatsu [12] also proposed that the Max-cut problems can be solved effectively by SOCP relaxation, and Zhang [18] provided the extension of a specific subclass of QOPs and generalized earlier results for finding optimal values. Unlike SOCP relaxation, LP relaxation is the most well-known for the fastest computing speed but has its weakness in obtaining tight bounds for an optimal value of the original QCQP.

The main purpose of this thesis is to solve nonconvex QCQPs where $\boldsymbol{Q}_i's$ are not positive semidefinite in (1) which is a NP-hard problem. We use the Assumption 4.1 in [11] where constraint matrices $\boldsymbol{Q}_i's$ are hollow matrices. A hollow matrix is a square matrix whose diagonal elements are all equal to zero. From this feature, we can yield a powerful conclusion in QCQP relaxations: $\zeta^*_{LP} = \zeta^*_{SOCP} = \zeta^*_{SDP} \leq \zeta^*$, where ζ^* is the optimal value of a QCQP that minimizes the objective function. The above equivalence means that the optimal values of SDP, SOCP, and LP are equivalent in nonconvex problems. Secondly, to solve problems which do not satisfy the Assumption 4.1, we change nonzero diagonal elements to off-diagonal to rearrange them as zeros to apply the above equivalence. From this approach, we can expect computational efficiency by creating a sparse matrix and can get reasonable approximated optimal solution similar to the original problem. In other words, with much less computational effort, the LP relaxation can be used to get the same quality of the optimal value instead of SDP relaxation.

This thesis is organized as follows: In Section 2, we introduce notation, and illustrate SDP, SOCP, and LP relaxations of QCQPs. In Section 3, we describe a specific example of QCQP, the pooling problem, and review the equivalent optimal values of the three relaxations based on [11]. In Section 4, we propose a rearranging method to deal with nonzero diagonal elements in detail. Section 5 presents numerical results on various problems. In Section 6, we present our conclusions.

2 Preliminaries

2.1 Notation and general QCQP

We denote the following notation throughout this thesis.

$\mathbb{R}^n$	:	The set of n-dimensional column vectors with real values
$\mathbb{R}^n_+$	:	The nonnegative orthant of $\mathbb{R}^n$
$\mathbb{R}^{m\times n}$	:	The set of m $\times$ n matrices with real values
$\mathbb{S}^n$	:	The set of real symmetric n $\times$ n matrices
$\mathbb{S}^n_+$	:	The set of n $\times$ n positive semidefinite matrices
$\boldsymbol{X} \succeq \boldsymbol{O}$	:	$\boldsymbol{X}$ is a positive semidefinite matrix
		$\Leftrightarrow$ $\boldsymbol{X} \in \mathbb{S}^n_+$ for n $\times$ n symmetric matrix X
		$\Leftrightarrow$ for $\forall \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T\boldsymbol{X}\boldsymbol{x} \geq \boldsymbol{0}$
$\boldsymbol{X} \bullet \boldsymbol{Y}$	:	The product of two equivalent dimensional matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$
		$\Leftrightarrow$ $\sum_{i=1}^n \sum_{j=1}^n \boldsymbol{X}_{ij}\boldsymbol{Y}_{ij}$
		$\Leftrightarrow < \boldsymbol{X}, \boldsymbol{Y} >$

We briefly discuss the structure of QCQP described in Section 1.

Let $\boldsymbol{x} \in \mathbb{R}^n$. The original form of QCQP is as follows:

$$\begin{aligned} \min_{x} \quad & \boldsymbol{x}^T\boldsymbol{Q}_0\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{q}_0 + r_0 \\ \text{subject to} \quad & \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{q}_i + r_i \leq 0 \quad (i = 1, ..., m\ ), \end{aligned} \tag{2}$$

where $\boldsymbol{Q}_i \in \mathbb{S}^n$, $\boldsymbol{q}_k \in \mathbb{R}^n$ $(i = 0, \ldots, m)$ and $\boldsymbol{r}_i \in \mathbb{R}$ $(i = 0, \ldots, m)$. Comparing (2) to the equation (1), we can consider the linear constraint $\boldsymbol{Ax} \leq \boldsymbol{b}$ as a part of the equation (2) when the $\boldsymbol{Q}_i$ is the zero matrix (i.e. every element in $\boldsymbol{Q}_i$ is zero).

We note that since $\boldsymbol{Q}_i \in \mathbb{S}^n$ $(i = 0, \ldots, m)$ is not necessarily positive semidefinite, (2) is a nonconvex problem.

2.2 SDP relaxation

From the general QCQP (2), we introduce a new variable matrix $\boldsymbol{X} \in \mathbb{S}^n$ as follows. Let

$$\bar{\boldsymbol{Q}}_i := \begin{pmatrix} \boldsymbol{r}_i & \boldsymbol{q}_i^T/2 \\ \boldsymbol{q}_i/2 & \boldsymbol{Q}_i \end{pmatrix}, \ \bar{\boldsymbol{X}} := \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix}, \text{ and } \bar{\boldsymbol{H}}_0 := \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix}.$$

Then, the equation (2) can be written as

$$\begin{aligned} \min_x \quad & \begin{pmatrix} \boldsymbol{r}_0 & \boldsymbol{q}_0^T/2 \\ \boldsymbol{q}_0/2 & \boldsymbol{Q}_0 \end{pmatrix} \bullet \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \\ \text{subject to} \quad & \begin{pmatrix} \boldsymbol{r}_i & \boldsymbol{q}_i^T/2 \\ \boldsymbol{q}_i/2 & \boldsymbol{Q}_i \end{pmatrix} \bullet \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \leq 0 \quad (i = 1, \ldots, m), \\ & \begin{pmatrix} \boldsymbol{1} & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} = 1. \end{aligned}$$

i.e.

$$\begin{aligned} \min_x \quad & \bar{\boldsymbol{Q}}_0 \bullet \bar{\boldsymbol{X}} \\ \text{subject to} \quad & \bar{\boldsymbol{Q}}_i \bullet \bar{\boldsymbol{X}} \leq 0 \qquad (i = 1, ..., m), \\ & \bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{X}} = 1. \end{aligned}$$

Proposition 1. *(the Schur Complement)*

Given $\boldsymbol{x} \in \mathbb{R}^n$ *and a matrix* $\boldsymbol{X} \in \mathbb{R}^{n \times n}$,

$$\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq 0 \quad \textit{if and only if} \quad \begin{pmatrix} \boldsymbol{1} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \succeq 0 \ .$$

By using the Schur Complement, the general form of QCQP (2) is transformed to SDP relaxation as follows:

$$
\begin{aligned}
\zeta^*_{SDP} := \min \quad & \bar{\boldsymbol{Q}}_0 \bullet \bar{\boldsymbol{X}} \\
\text{subject to} \quad & \bar{\boldsymbol{Q}}_i \bullet \bar{\boldsymbol{X}} \leq 0 \quad (i = 1, \ldots, m), \\
& \bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{X}} = 1, \\
& \bar{\boldsymbol{X}} \in \mathbb{S}^{n+1}_+ .
\end{aligned} \tag{3}
$$

2.3 SOCP relaxation

The SOCP relaxation is described as follows:

$$
\left.
\begin{aligned}
\min \quad & \bar{\boldsymbol{Q}}_0 \bullet \bar{\boldsymbol{X}} \\
\text{subject to} \quad & \bar{\boldsymbol{Q}}_i \bullet \bar{\boldsymbol{X}} \leq 0 \ (1 \leq i \leq m), \ \ \bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{X}} = 1, \\
& \bar{X}_{jj} \geq 0 \ (1 \leq j \leq n+1), \\
& (\bar{X}_{ij})^2 \leq \bar{X}_{ii}\bar{X}_{jj} \ (1 \leq i < j \leq n+1) \ .
\end{aligned}
\right\} \tag{4}
$$

Definition 1. *(Second Order Cone)*

A Second Order Cone (SOC) is by definition a cone of the form

$$SOC := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{N-1} \mid x_1 \geq ||x_2||\},$$

where $\| \centerdot \|$ *denotes the Euclidean norm. The SOC is also known as the quadratic cone or Lorentz cone.*

Definition 2. *(Second Order Cones) We know*

$w^2 \leq \xi\eta,\ \xi \geq 0$ *and* $\eta \geq 0$ *if and only if* $\left\| \begin{pmatrix} \xi - \eta \\ 2w \end{pmatrix} \right\| \leq \xi + \eta.$

Then, we see that $\begin{pmatrix} \xi + \eta \\ \xi - \eta \\ 2w \end{pmatrix} \in$ *Second Order Cone* (SOC).

Kim and Kojima [15] used the characteristic of $\bar{\boldsymbol{X}} \in \mathbb{S}_{+}^{n+1}$ in the equation (3) to derive second order cone (SOC) constraints. Since "every 2 by 2 submatrices in $\bar{\boldsymbol{X}}$ is positive semidefinite" is equivalent to $\bar{X}_{ii}\bar{X}_{jj} - \bar{X}_{ij}\bar{X}_{ji} \geq 0 \ for \ \forall i$ and $\forall j$, we can rewrite this relation as $(\bar{X}_{ij})^2 \leq \bar{X}_{ii}\bar{X}_{jj}$ $(1 \leq i < j \leq n+1)$ where $\bar{X}_{ij} = \bar{X}_{ji}$ because the matrix $\boldsymbol{X}$ is symmetric. Therefore, by the definition (2), we can convert (4) to the following equivalent problem:

$$
\left.\begin{array}{lll}
\zeta_{SOCP}^{*} := \min & \bar{\boldsymbol{Q}}_0 \bullet \bar{\boldsymbol{X}} & \\
\text{subject to} & \bar{\boldsymbol{Q}}_i \bullet \bar{\boldsymbol{X}} \leq 0 \ (1 \leq i \leq m), \ \bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{X}} = 1, & \\
& \left\| \begin{pmatrix} \bar{X}_{ii} - \bar{X}_{jj} \\ 2\bar{X}_{ij} \end{pmatrix} \right\| \leq \bar{X}_{ii} + \bar{X}_{jj}, & \\
& (1 \leq i < j \leq n+1), & \\
& where \ \bar{X}_{ij} = \bar{X}_{ji} \ for \ \forall i \text{ and } \ \forall j. &
\end{array}\right\} \tag{5}
$$

The SOCP relaxation (5) has been studied in many applications due to its computational efficiency in solving QCQPs. More recently, it has been employed to solve polynomial optimization problems. In particular, (5) appeared in the hierarchy of the scaled diagonally dominant sum-of-squares (SDSOS) relaxations proposed in [1]:

$$DSOS \subseteq SDSOS \subseteq SOS \subseteq PSD,$$

where SOS is a cone of sum-of-squares and DSOS is a cone of diagonally dominant sum-of-squares. The paper [1] also proposed DSOS and SDSOS can be done with Linear Programming and Second Order Cone Programming respectively. Although DSOS and SDSOS relaxations are not as precise as SOS, they can be alternatives to SOS optimization because of the computational advantages.

2.4 LP relaxation

The LP relaxation can be represented as follows:

$$\left.\begin{array}{ll} \zeta_{LP}^* := \min & \bar{\boldsymbol{Q}}_0 \bullet \bar{\boldsymbol{X}} \\ \text{subject to} & \bar{\boldsymbol{Q}}_i \bullet \bar{\boldsymbol{X}} \leq 0 \ (1 \leq i \leq m),\ \bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{X}} = 1, \\ & \bar{X}_{ii} \geq 0 \ (1 \leq i \leq n+1), \\ & \bar{X}_{ii} + \bar{X}_{jj} - 2|\bar{X}_{ij}| \geq 0 \\ & (1 \leq i < j \leq n+1), \\ & where\ \bar{X}_{ij} = \bar{X}_{ji}\ for\ \forall i\ and\ \forall j\ . \end{array}\right\} \tag{6}$$

The problem (6) is derived from the diagonally dominant sum-of-squares relaxation (DSOS) in [1]. Consider the set of diagonally dominant matrices of dimension $n+1$:

$$\boldsymbol{D}^{n+1} := \left\{ \boldsymbol{X} \in \mathbb{S}^{n+1} : X_{ii} \geq \sum_{j \neq i} |X_{ij}| \quad (1 \leq i \leq n+1) \right\}.$$

In [6], the dual of $\boldsymbol{D}^{n+1}$ is given by

$$\begin{aligned} (\boldsymbol{D}^{n+1})^* &:= \left\{ \boldsymbol{X} \in \mathbb{S}^{n+1} : \boldsymbol{x}^T \boldsymbol{X} \boldsymbol{x} \geq 0 \text{ for } \forall \boldsymbol{x} \text{ with at most 2 elements } \pm 1 \right\} \\ &= \left\{ \boldsymbol{X} \in \mathbb{S}^{n+1} : X_{ii} \geq 0,\ X_{ii} + X_{jj} - 2|X_{ij}| \geq 0 \quad (1 \leq i < j \leq n+1) \right\}. \end{aligned}$$

Let $\bar{\boldsymbol{X}}$ be a feasible solution of the SOCP relaxation (4). Then,

$$|\bar{X}_{ij}| \leq \sqrt{\bar{X}_{ii}\bar{X}_{jj}} \ \ (1 \leq i < j \leq n+1).$$

Since $\sqrt{\bar{X}_{ii}\bar{X}_{jj}} \leq (\bar{X}_{ii} + \bar{X}_{jj})/2$ always holds for all nonnegative $\bar{X}_{ii}$ and $\bar{X}_{jj}$, $\bar{\boldsymbol{X}}$ is a feasible solution of (6). Thus, the LP relaxation (6) is an weaker relaxation than the SOCP relaxation (5). In conclusion, we can derive the relationship among the optimal values of the three relaxations:

$$\zeta_{LP}^* \leq \zeta_{SOCP}^* \leq \zeta_{SDP}^* \leq \zeta^*. \tag{7}$$

3 The equivalence among the SDP, SOCP, and LP relaxations

In Section 3.1, we describe the pooling problem, and the equivalence of the three relaxations under the assumption 4.1 [11] in Section 3.2 .

3.1 The pooling problem

In petroleum refinery industry, the pooling problem is used for measuring the minimum cost of gas flow. This problem is a nonlinear and nonconvex problem. Nishi [19] applied the SDP relaxation to the pooling problem, and Kimizuka et al. [11] showed the equivalence of the three relaxations and proposed the rescheduling method.

The pooling problem [4] deals with material blending by mixing crude or refined oil. In a storage tank, pooling occurs when streams are mixing together. There exist many routes from a tank to another tank, and the final storage tank dispatches the resulting mixture to several locations. Since the total cost depends on which route will be used, deriving the optimal route with maximum efficiency is the issue for companies. Blending various kinds of petroleum is therefore an important process to obtain end-products with quality specifications.

We can formulate this industry problem with several parameters, variables, and constants. The principal parameters are the number of sources, mixers, plants, and in-out nodes. The main variables are the flow of pipelines, amount and quality of materials, and compensation cost. The core constants are amount of raw materials, transportation costs, and min-max volume of mixers.

The formulation of the pooling problem proposed in [11] added a penalty term using δ in the objective function to have an interior point and to avoid numerical difficulties. The reformulated pooling problem is as follows:

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^n,\ \boldsymbol{\lambda}\in\mathbb{R}^{m+d}} \quad & \boldsymbol{q}_0^T\boldsymbol{x} + \delta\sum_{i=1}^{m+d}\lambda_i \\
\text{subject to} \quad & \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + \boldsymbol{q}_i^T\boldsymbol{x} - \lambda_i^1 + \gamma_i \le 0, \\
& -\boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} - \boldsymbol{q}_i^T\boldsymbol{x} - \lambda_i^1 - \gamma_i \le 0 \ (i = 1,\ldots,m), \\
& \boldsymbol{q}_{m+j}\boldsymbol{x} - (\boldsymbol{\lambda}_{m+1}^2)_j - (\boldsymbol{b}_{m+1})_j \le 0, \\
& -\boldsymbol{q}_{m+j}\boldsymbol{x} - (\boldsymbol{\lambda}_{m+1}^2)_j + (\boldsymbol{b}_{m+1})_j \le 0 \ (j = 1,\ldots,d), \\
& \boldsymbol{q}_{m+j+k}\boldsymbol{x} - (\boldsymbol{b}_{m+2})_k \le 0 \ (k = 1,\ldots,e), \\
& \boldsymbol{\lambda} \ge 0\ ,\ \boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u},
\end{aligned} \tag{8}
$$

where m, d, and e are the numbers of each inequality constraints, $\boldsymbol{l}$ and $\boldsymbol{u}$ are the lower and upper bound for $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ respectively, $\boldsymbol{b}$ and $\boldsymbol{\gamma}$ are constraints. Notice that the diagonal of the coefficient matrices of (8), i.e. the diagonal of $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_m$ in the quadratic constraints, are zeros. As mentioned previously, these matrices are hollow matrices. This property is used in Section 3.2 to prove the equivalence of the optimal values of the three relaxation methods.

3.2 The equivalence of the optimal values of the three relaxations

The equivalence of the optimal values of the SDP, SOCP and LP relaxations is derived for QCQPs where the diagonal elements of the coefficient matrices in (2) are all zeros. To introduce this property, the relationship between primal and dual approaches [14] is used. Moreover, the results from the diagonally dominant sums of squares relaxations (DSOS) and scaled diagonally dominant sums of squares (SD-SOS) for the hierarchies among the SDP, SOCP, and LP relaxations [1] were exploited in [11].

Theorem 1. *(Weak Duality, [14]) Let* ζ^* *and* μ^* *be optimal values of primal and dual problems respectively. Then* $\mu^* \leq \zeta^*$ *.*

The weak duality can be derived from the Lagrangian function, which yields weak duality of the optimal values in primal and dual problem. Even when the original problem is not convex, this inequality always holds. For more information, we refer to [14].

Theorem 2. *[2] If* ζ^*_{SDP}*,* ζ^*_{SOCP}*, and* ζ^*_{LP} *are the optimal values of the SDP, SOCP, and LP relaxations respectively, then* $\zeta^*_{LP} \leq \zeta^*_{SOCP} \leq \zeta^*_{SDP}$ *.*

From Theorem 2, we can easily derive the relationship among the dual optimal values, $\mu^*_{SDP} \leq \mu^*_{SOCP} \leq \mu^*_{LP}$. In [11], they used Theorem 2 along with the Schur Complement and eigenvalue properties to lead $\zeta^*_{SDP} \leq \zeta^*_{LP}$ under certain assumption: all the diagonal elements in $\boldsymbol{Q}_0, \boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m$ of (2) are zeros. The following theorem is an intermediate conclusion of their work [11]. We include their results in the following for completeness.

Theorem 3. *[11] If all the diagonal elements in* $\boldsymbol{Q}_0, \boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m$ *of* (2) *are zeros, then* $\zeta^*_{SDP} = \zeta^*_{SOCP} = \zeta^*_{LP}$ *.*

Proof. We will prove $\zeta^*_{SDP} \leq \zeta^*_{LP}$. Let $\bar{\boldsymbol{X}} = \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix}$ be a feasible solution of (6). From the constraint $\bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{W}} = 1$, we can say $x_{00} = 1$. Therefore, $\bar{\boldsymbol{X}} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix}$. Now let α be a constant which is greater than or equal to the maximum eigenvalue of the matrix $\boldsymbol{x}\boldsymbol{x}^T - \boldsymbol{X}$. In other words, $\alpha \geq \lambda_{\max}\left(\boldsymbol{x}\boldsymbol{x}^T - \boldsymbol{X}\right)$. If we add α to the diagonal of $\bar{\boldsymbol{X}}$ except the first diagonal element x_{00} of $\bar{\boldsymbol{X}}$, the resulting matrix $\begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} + \alpha \boldsymbol{I} \end{pmatrix}$ becomes positive semidefinite by the Schur complement, from

$$\boldsymbol{X} + \alpha\boldsymbol{I} - \boldsymbol{x}\boldsymbol{x}^T = (\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T) + \alpha\boldsymbol{I} = \lambda^*\boldsymbol{I} + \alpha\boldsymbol{I} = (\lambda^* + \alpha)\boldsymbol{I},$$

where $\lambda^* \leq \alpha$ since α is greater than or equal to the maximum eigenvalue of $\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T$. From the above eigenvalue inequality, $\lambda^* + \alpha \geq 0$ always holds, and we can say that $\boldsymbol{X} + \alpha\boldsymbol{I} - \boldsymbol{x}\boldsymbol{x}^T \succeq 0$. Therefore, by the Schur Complement, the resulting matrix $\begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} + \alpha\boldsymbol{I} \end{pmatrix}$ is positive semidefinite.

Note that the inequality constraints still hold and the objective value remains same since the diagonal elements in $\boldsymbol{Q}_0, \ldots, \boldsymbol{Q}_m$ are zeros by the assumption. Thus, $\begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} + \alpha\boldsymbol{I} \end{pmatrix}$ is a feasible solution of the SDP relaxation (3). Thus, we can construct a feasible solution in the SDP relaxation whose objective value is same as $\bar{\boldsymbol{X}}$. This fact leads to $\zeta^*_{SDP} \leq \zeta^*_{LP}$, and therefore $\zeta^*_{SDP} = \zeta^*_{SOCP} = \zeta^*_{LP}$ follows. $\square$

Theorem 3 indicates that the primal optimal values of the SOCP, LP, and SDP relaxations are computationally the same if all the diagonal elements of coefficient matrices are zeros. Now, we need to prove the equivalence between primal and dual values among the three relaxations using the SOS hierarchy [1] to obtain $\zeta^*_{SDP} = \zeta^*_{SOCP} = \zeta^*_{LP} = \mu^*_{LP} = \mu^*_{SOCP} = \mu^*_{SDP}$.

First, the dual of SDP (3), SOCP (5), and LP (6) relaxations can be written as follows, respectively by the Lagrangian duality theorem [14]:

$$\begin{aligned} \mu^*_{SDP} := \max \quad & \mu \\ \text{subject to} \quad & \bar{\boldsymbol{Q}}_0 + \textstyle\sum_{k=1}^m \eta_k \bar{\boldsymbol{Q}}_k - \mu\bar{\boldsymbol{H}}_0 - \bar{\boldsymbol{Y}} = \boldsymbol{O}, \\ & \eta_1, \ldots, \eta_m \geq 0, \ \mu \in \mathbb{R}, \\ & \bar{\boldsymbol{Y}} \in \mathbb{S}^{n+1}_+, \end{aligned} \tag{9}$$

$$
\begin{aligned}
\mu^*_{SOCP} := \max \quad & \mu \\
\text{subject to} \quad & \bar{\boldsymbol{Q}}_0 + \textstyle\sum_{k=1}^m \eta_k \bar{\boldsymbol{Q}}_k - \mu \bar{\boldsymbol{H}}_0 - \bar{\boldsymbol{Y}} = \boldsymbol{O}, \\
& \eta_1, \ldots, \eta_m \geq 0, \ \ \mu \in \mathbb{R}, \\
& \bar{\boldsymbol{Y}} \in \boldsymbol{SD}^{n+1},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\mu^*_{LP} := \max \quad & \mu \\
\text{subject to} \quad & \bar{\boldsymbol{Q}}_0 + \textstyle\sum_{k=1}^m \eta_k \bar{\boldsymbol{Q}}_k - \mu \bar{\boldsymbol{H}}_0 - \bar{\boldsymbol{Y}} = \boldsymbol{O}, \\
& \eta_1, \ldots, \eta_m \geq 0, \ \ \mu \in \mathbb{R}, \\
& \bar{\boldsymbol{Y}} \in \boldsymbol{D}^{n+1},
\end{aligned}
\tag{11}
$$

where $\boldsymbol{SD}^{n+1}$ and $\boldsymbol{D}^{n+1}$ denote a cone of scaled diagonally dominant matrices (SDD) and a cone of diagonally dominant matrices (DD) respectively. We note that η_i $(i = 0, 1, \ldots, m)$ is the Lagrangian constraint to derive dual functions.

These dual problems are closely related to SDSOS and DSOS [1]. Particularly, in [2], SDSOS relaxations were proposed using SDD matrices, and showed that $\boldsymbol{D}^{n+1} \subset \boldsymbol{\mathcal{S}D}^{n+1} \subset \mathbb{S}^{n+1}_+$ holds. From the relations on $\boldsymbol{D}^{n+1}$ and $\boldsymbol{\mathcal{S}D}^{n+1}$, we see that $\mu^*_{SDP} \geq \mu^*_{SOCP} \geq \mu^*_{LP}$ is obtained. Also, since $\mu^*_{SDP} \leq \mu^*_{SOCP} \leq \mu^*_{LP}$ always holds by the Lagrangian duality theorem [14], $\mu^*_{SDP} = \mu^*_{SOCP} = \mu^*_{LP}$ holds by Theorem 3.

Corollary 3.1. *[11] If all the diagonal elements in $\boldsymbol{Q}_0, \boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m$ of* (2) *are zeros, it holds that* $\zeta^*_{SDP} = \zeta^*_{SOCP} = \zeta^*_{LP} = \mu^*_{LP} = \mu^*_{SOCP} = \mu^*_{SDP}$.

Proof. We will prove $\zeta^*_{LP} = \mu^*_{LP}$. Let $\bar{\boldsymbol{Y}} = \begin{pmatrix} y_{00} & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix}$ be a value in (9). Since the diagonal elements in $\bar{\boldsymbol{Q}}_i$ and $\bar{\boldsymbol{H}}_i$ (i = 1, …, m) are all zeros except (1,1) in (9) by the assumption, and by the Lagrangian duality theorem [14], the diagonal of $\boldsymbol{Y}$ should be zero. Since $\boldsymbol{Y} \in \mathbb{S}^n_+$, we have $\boldsymbol{Y} = \boldsymbol{O}$, thus, $\bar{\boldsymbol{Y}} \in \mathbb{S}^{n+1}_+$ leads to $\boldsymbol{y} = \boldsymbol{0}$. This is because the determinant of 2×2 submatrix of semidefinite matrix $\boldsymbol{Y}$ should be greater than or equal to zero. Hence, (9) is equivalent to the following problem:

$$
\begin{array}{ll}
\max & \mu \\
\text{subject to} & \gamma_0 + \sum_{k=1}^{m} \eta_k \gamma_k - \mu - y_{00} \geq 0, \\
& \boldsymbol{q}_0 + \sum_{k=1}^{m} \eta_k \boldsymbol{q}_k = \boldsymbol{0}, \\
& \boldsymbol{Q}_0 + \sum_{k=1}^{m} \eta_k \boldsymbol{Q}_k = \boldsymbol{O}, \\
& \eta_1, \ldots, \eta_m \geq 0,\ \mu \in \mathbb{R},\ y_{00} \geq 0.
\end{array} \tag{12}
$$

Likewise, for (11), we can show that $\boldsymbol{Y} = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{0}$ using the zero diagonal of $\bar{\boldsymbol{Y}}$. As a result, (11) is equivalent to (12), and the optimal values of (6) and (12) coincide, i.e., $\mu^*_{SDP} = \mu^*_{LP}$. Since the duality theorem holds on the LP relaxation regardless of the existence of interior points, the optimal values of (6) and (11) are equivalent, *i.e.*, $\zeta^*_{LP} = \mu^*_{LP}$. By $\mu^*_{SDP} = \mu^*_{LP} = \zeta^*_{LP}$, $\mu^*_{SDP} \geq \mu^*_{SOCP} \geq \mu^*_{LP}$, (7), and Theorem 3, the desired result follows. □

Finally, it was shown in [11] that $\zeta^*_{LP} = \mu^*_{LP}$ to satisfy Corollary 3 using the existence of interior point in the primal and dual LP problems. Since $\mu^*_{LP} = \mu^*_{SOCP} = \mu^*_{SDP}$ and $\zeta^*_{SDP} = \zeta^*_{SOCP} = \zeta^*_{LP}$ hold, we conclude that the primal and dual optimal values of the SDP, SOCP, and LP relaxations are equivalent by Theorem 3.

4 Proposed method

We propose our method to solve QCQP whose coefficient matrices have nonzero diagonal elements. Our proposed method is based on the theoretical properties of QCQP with the hollow coefficient matrices. More precisely, we convert QCQPs with nonzero diagonal coefficient matrices to QCQPs with the hollow coefficient matrices by introducing new variables. Then, we apply the results described in Section 3 to solve the transformed QCQP with numerically efficient SOCP and LP relaxations, instead of SDP relaxation.

4.1 Rearrangement of diagonal elements

Assume that there exists at least one nonzero diagonal element in $\boldsymbol{Q}_i$ for some i of the general QCQP (2). To use theorem 3, we remove diagonal elements in $\boldsymbol{Q}_i$ and introduce an off-diagonal and a new equality constraint to make the diagonal zero. First, let us focus on the original matrix of $\boldsymbol{x}\boldsymbol{x}^T$:

$$\begin{aligned} \boldsymbol{x}\boldsymbol{x}^T &= [\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n]^T \times [\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n] \\ &= \boldsymbol{X} \end{aligned}$$

Then $\bar{\boldsymbol{X}}$ can be written as follows:

$$\bar{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{x}_{00} & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \\ \boldsymbol{x}_1 & \boldsymbol{x}_1^2 & \boldsymbol{x}_1\boldsymbol{x}_2 & \dots & \boldsymbol{x}_1\boldsymbol{x}_n \\ \boldsymbol{x}_2 & \boldsymbol{x}_2\boldsymbol{x}_1 & \boldsymbol{x}_2^2 & \dots & \boldsymbol{x}_2\boldsymbol{x}_n \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{x}_n & \boldsymbol{x}_n\boldsymbol{x}_1 & \boldsymbol{x}_n\boldsymbol{x}_2 & \dots & \boldsymbol{x}_n^2 \end{pmatrix} = \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix},$$

where $\boldsymbol{x}_{00} = 1$.

Then, $\bar{\boldsymbol{Q}}_i \bullet \bar{\boldsymbol{X}}$ in the nonconvex QCQP can be written as:

$$\begin{pmatrix} \boldsymbol{r}_i & \boldsymbol{q}_{i1} & \boldsymbol{q}_{i2} & \dots & \boldsymbol{q}_{in} \\ \boldsymbol{q}_{i1} & \boldsymbol{Q}_{i_{11}} & \boldsymbol{Q}_{i_{12}} & \dots & \boldsymbol{Q}_{i_{1n}} \\ \boldsymbol{q}_{i2} & \boldsymbol{Q}_{i_{21}} & \boldsymbol{Q}_{i_{22}} & \dots & \boldsymbol{Q}_{i_{2n}} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{q}_{in} & \boldsymbol{Q}_{i_{n1}} & \boldsymbol{Q}_{i_{n2}} & \dots & \boldsymbol{Q}_{i_{nn}} \end{pmatrix} \bullet \begin{pmatrix} \boldsymbol{x}_{00} & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \\ \boldsymbol{x}_1 & \boldsymbol{x}_1^2 & \boldsymbol{x}_1\boldsymbol{x}_2 & \dots & \boldsymbol{x}_1\boldsymbol{x}_n \\ \boldsymbol{x}_2 & \boldsymbol{x}_2\boldsymbol{x}_1 & \boldsymbol{x}_2^2 & \dots & \boldsymbol{x}_2\boldsymbol{x}_n \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{x}_n & \boldsymbol{x}_n\boldsymbol{x}_1 & \boldsymbol{x}_n\boldsymbol{x}_2 & \dots & \boldsymbol{x}_n^2 \end{pmatrix},$$

where $\boldsymbol{Q}_{i_{jj}}$ is not zero for some j. We relax $\bar{\boldsymbol{X}}$ using the Schur Complement so that the elements in a new matrix $\bar{\boldsymbol{X}}$ become linear variables in the relaxed problems. More precisely,

$$\begin{pmatrix} \boldsymbol{x}_{00} & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \\ \boldsymbol{x}_1 & \boldsymbol{x}_1^2 & \boldsymbol{x}_1\boldsymbol{x}_2 & \dots & \boldsymbol{x}_1\boldsymbol{x}_n \\ \boldsymbol{x}_2 & \boldsymbol{x}_2\boldsymbol{x}_1 & \boldsymbol{x}_2^2 & \dots & \boldsymbol{x}_2\boldsymbol{x}_n \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{x}_n & \boldsymbol{x}_n\boldsymbol{x}_1 & \boldsymbol{x}_n\boldsymbol{x}_2 & \dots & \boldsymbol{x}_n^2 \end{pmatrix} \Rightarrow \begin{pmatrix} \bar{\boldsymbol{X}}_{1,1} & \bar{\boldsymbol{X}}_{1,2} & \bar{\boldsymbol{X}}_{1,3} & \dots & \bar{\boldsymbol{X}}_{1,n+1} \\ \bar{\boldsymbol{X}}_{2,1} & \bar{\boldsymbol{X}}_{2,2} & \bar{\boldsymbol{X}}_{2,3} & \dots & \bar{\boldsymbol{X}}_{2,n+1} \\ \bar{\boldsymbol{X}}_{3,1} & \bar{\boldsymbol{X}}_{3,2} & \bar{\boldsymbol{X}}_{3,3} & \dots & \bar{\boldsymbol{X}}_{3,n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \bar{\boldsymbol{X}}_{n+1,1} & \bar{\boldsymbol{X}}_{n+1,2} & \bar{\boldsymbol{X}}_{n+1,3} & \dots & \bar{\boldsymbol{X}}_{n+1,n+1} \end{pmatrix}.$$

From this SDP relaxation, we regard $\bar{\boldsymbol{X}}_{i,j}$ as a new variable in the relaxation problem of the original QCQP.

Second, for simplicity, we assume $\boldsymbol{Q}_{i_{jj}}$ is the only one nonzero diagonal element of $\boldsymbol{Q}_i$ in QCQP. To apply Theorem 3 and Corollary 3.1 , all the diagonal elements in $\boldsymbol{Q}_0, \boldsymbol{Q}_1, \dots, \boldsymbol{Q}_m$ should be zeros. Thus, we rearrange $\boldsymbol{Q}_{i_{jj}}$ which is the coefficient element for $\boldsymbol{x}_j^2$ by adding a new variable. Let $\boldsymbol{x}_l$ be a new variable to satisfy $\boldsymbol{x}_j^2 = \boldsymbol{x}_j\boldsymbol{x}_j = \boldsymbol{x}_j\boldsymbol{x}_l$ where $\boldsymbol{x}_j = \boldsymbol{x}_l$. Then we can move $\boldsymbol{Q}_{i_{jj}}$ located in (j, j) of matrix $\boldsymbol{Q}_i$ to $(j+1, (1+n)+1)$ and $((1+n)+1, j+1)$th positions by making new matrix $\bar{\boldsymbol{Q}}_i{}^*$. More precisely, we replace $\boldsymbol{Q}_{i_{jj}}$ by $\frac{\bar{\boldsymbol{Q}}^*_{i_{j+1,n+2}}}{2} + \frac{\bar{\boldsymbol{Q}}^*_{i_{n+2,j+1}}}{2}$ in the symmetric matrix $\bar{\boldsymbol{Q}}_i^*$ to make $\boldsymbol{Q}_{i_{jj}}$ as zero. In other words,

$$\boldsymbol{Q}_{i_{jj}} \times \boldsymbol{x}_j^2 = \frac{\bar{\boldsymbol{Q}}^*_{i_{j+1,n+2}}}{2} \times \boldsymbol{x}_j\boldsymbol{x}_l + \frac{\bar{\boldsymbol{Q}}^*_{i_{n+2,j+1}}}{2} \times \boldsymbol{x}_l\boldsymbol{x}_j.$$

As a result, we can write $\bar{\boldsymbol{Q}}_i{}^*$ as follows:

$$\bar{\boldsymbol{Q}}_i{}^* = \begin{pmatrix} \boldsymbol{r}_i & \boldsymbol{q}_{i1} & \boldsymbol{q}_{i2} & \dots & \boldsymbol{q}_{ij} & \dots & \boldsymbol{q}_{in} & 0 \\ \boldsymbol{q}_{i1} & 0 & \boldsymbol{Q}_{i_{12}} & \dots & \boldsymbol{Q}_{i_{1j}} & \dots & \boldsymbol{Q}_{i_{1n}} & 0 \\ \boldsymbol{q}_{i2} & \boldsymbol{Q}_{i_{21}} & 0 & \dots & \boldsymbol{Q}_{i_{2j}} & \dots & \boldsymbol{Q}_{i_{2n}} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \boldsymbol{q}_{ij} & \boldsymbol{Q}_{i_{j1}} & \boldsymbol{Q}_{i_{j2}} & \dots & 0 & \dots & \boldsymbol{Q}_{i_{jn}} & \frac{\boldsymbol{Q}_{i_{jj}}}{2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \boldsymbol{q}_{in} & \boldsymbol{Q}_{i_{n1}} & \boldsymbol{Q}_{i_{n2}} & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\boldsymbol{Q}_{i_{jj}}}{2} & 0 & 0 & 0 \end{pmatrix},$$

where l = n+1 and 0 is a zero element.

Now, we rewrite $\bar{\boldsymbol{X}}$ and $\bar{\boldsymbol{H}}_0$ to be consistent with the size of $\bar{\boldsymbol{Q}}_i{}^*$ by adding a new variable $\boldsymbol{x}_{n+1}$:

$$\begin{aligned} \boldsymbol{x}^* &= [\mathbf{1}, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_n, \boldsymbol{x}_{n+1}] \\ \bar{\boldsymbol{X}}^* &= [\mathbf{1}, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_n, \boldsymbol{x}_{n+1}]^T \times [\mathbf{1}, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_n, \boldsymbol{x}_{n+1}] \in \mathbb{S}_+^{n+2} \\ \boldsymbol{H}_0^* &= \text{The zero matrix except the first element which is } 1 \in \mathbb{S}^{n+2}. \end{aligned}$$

The sizes of $\bar{\boldsymbol{X}}^*$ and $\bar{\boldsymbol{Q}}_i{}^*$ have become $(n+2) \times (n+2)$, but the diagonal elements in $\bar{\boldsymbol{Q}}_i{}^*$ are all zeros except γ_i. Thus, we can apply Corollary 3.1, and can expect the same optimal values by solving the modified SDP, SOCP, and LP relaxations of the transformed QCQP. We should mention that $\bar{\boldsymbol{Q}}_i{}^*$ is a sparse matrix because the $(n+2)$th column and the row are all zeros except two elements in $(j+1, (1+n)+1)$ and $((1+n)+1, j+1)$ with the value $\frac{\boldsymbol{Q}_{i_{jj}}}{2}$.

For $k = n+1$, a new SDP relaxation is as follows:

$$
\begin{aligned}
\zeta^*_{SDP_{new}} := \min \quad & \bar{\boldsymbol{Q}}^*_0 \bullet \bar{\boldsymbol{X}}^* \\
\text{subject to} \quad & \bar{\boldsymbol{Q}}^*_i \bullet \bar{\boldsymbol{X}}^* \le 0 \quad (i = 1, \dots, m), \\
& \bar{\boldsymbol{H}}^*_0 \bullet \bar{\boldsymbol{X}}^* = 1, \\
& \bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,n+2}, \quad \bar{X}^*_{j+1,j+1} = \bar{X}^*_{n+2,j+1}\ , \\
& \bar{\boldsymbol{X}}^* \in \mathbb{S}^{n+2}_+,
\end{aligned}
$$

where $\bar{X}^*_{j+1,j+1}$, $\bar{X}^*_{j+1,n+2}$, and $\bar{X}^*_{n+2,j+1}$ are relaxed elements of $\boldsymbol{x}_j^2$, $\boldsymbol{x}_j\boldsymbol{x}_{n+1}$, and $\boldsymbol{x}_{n+1}\boldsymbol{x}_j$, respectively.

Third, if the number of nonzero elements are more than one, the number of the nonzero diagonal elements can be at most n when the size of $\boldsymbol{Q}_i's$ in the original QCQP is n. Thus, the size of a new matrix $\bar{\boldsymbol{Q}}^*_i$ can increase up to $\boldsymbol{1+2n}$. Let $\mathcal{C}$ denote the vector which contains the location of nonzero diagonal elements in every $\boldsymbol{Q}_i's$ of general QCQP (2). Then the length of $\mathcal{C}$ can be from 1 to $2n$ if there exists at least one nonzero element in the diagonal. For instance, if $(2,2)$ is not zero in $\boldsymbol{Q}_1$, and $(1,1), (2,2), (3,3)$ and $(5,5)$ are not zeros in $\boldsymbol{Q}_2$, $\mathcal{C}$ can be written as $[1,2,3,5]$. In this case, four nonzero diagonal elements yield four new variables, $\boldsymbol{x}_{n+1}$, $\boldsymbol{x}_{n+2}$, $\boldsymbol{x}_{n+3}$ and $\boldsymbol{x}_{n+4}$. Thus, We can rewrite the SDP problem in a new general form if the number of nonzero diagonal elements is $k(=\text{length}(\mathcal{C}))$:

$$
\begin{aligned}
\zeta^*_{SDP_{new}} := \min \quad & \bar{\boldsymbol{Q}}^*_0 \bullet \bar{\boldsymbol{X}}^* \\
\text{subject to} \quad & \bar{\boldsymbol{Q}}^*_i \bullet \bar{\boldsymbol{X}}^* \le 0 \quad (i = 1, \dots, m), \\
& \bar{\boldsymbol{H}}^*_0 \bullet \bar{\boldsymbol{X}}^* = 1, \\
& \bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,\ 1+n+p} \\
& p = 1, \dots, k, \quad j = \mathcal{C}(p), \\
& \bar{\boldsymbol{X}}^* \in \mathbb{S}^{1+n+k}_+,
\end{aligned}
\tag{13}
$$

where $\bar{\boldsymbol{Q}}^*_i \in \mathbb{S}^{1+n+k}$, *and* $\boldsymbol{H}_0 \in \mathbb{S}^{1+n+k}$.

Note that we have added linear constraints $\bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,\ 1+n+p}$ in (13) without other constraints such as $\bar{X}^*_{i,j+1} = \bar{X}^*_{i,\ 1+n+p}\ \forall i$, $\bar{X}^*_{j+1,\ i} = \bar{X}^*_{1+n+p,i}\ \forall i$ and $\bar{X}^*_{j+1,\ j+1} = \bar{X}^*_{1+n+p,\ 1+n+p}$ because $\boldsymbol{x}_j^2 = \boldsymbol{x}_j \boldsymbol{x}_l$ connotes $\boldsymbol{x}_j = \boldsymbol{x}_l$ and $\boldsymbol{x}_j^2 = \boldsymbol{x}_l^2$ where $\boldsymbol{x}_j$ is nonnegative by the assumption. Moreover, since $\bar{\boldsymbol{X}}^*$ is symmetric, we do not need $\bar{X}^*_{j+1,j+1} = \bar{X}^*_{1+n+p,\ j+1}$. As a result, $\bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,\ 1+n+p}$ contains all of the above additional linear constraints, and so we can expect the same computational result from the new SDP relaxation.

We present a description of general matrix form of $\bar{\boldsymbol{Q}}_i^*$ if there exist n number of nonzero diagonal elements:

$$\begin{pmatrix}
\boldsymbol{r}_i & \boldsymbol{q}_{i1} & \boldsymbol{q}_{i2} & \dots & \boldsymbol{q}_{ij} & \dots & \boldsymbol{q}_{in} & 0 & 0 & 0 & 0 & 0 & 0 \\
\boldsymbol{q}_{i1} & 0 & \boldsymbol{Q}_{i_{12}} & \dots & \boldsymbol{Q}_{i_{1j}} & \dots & \boldsymbol{Q}_{i_{1n}} & \frac{\boldsymbol{Q}_{i_{11}}}{2} & 0 & 0 & 0 & 0 & 0 \\
\boldsymbol{q}_{i2} & \boldsymbol{Q}_{i_{21}} & 0 & \dots & \boldsymbol{Q}_{i_{2j}} & \dots & \boldsymbol{Q}_{i_{2n}} & 0 & \frac{\boldsymbol{Q}_{i_{22}}}{2} & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\boldsymbol{q}_{ij} & \boldsymbol{Q}_{i_{j1}} & \boldsymbol{Q}_{i_{j2}} & \dots & 0 & \dots & \boldsymbol{Q}_{i_{jn}} & \mathbf{0} & 0 & 0 & \frac{\boldsymbol{Q}_{i_{jj}}}{2} & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\boldsymbol{q}_{in} & \boldsymbol{Q}_{i_{n1}} & \boldsymbol{Q}_{i_{n2}} & \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\boldsymbol{Q}_{i_{nn}}}{2} \\
0 & \frac{\boldsymbol{Q}_{i_{11}}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\boldsymbol{Q}_{i_{22}}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\boldsymbol{Q}_{i_{jj}}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\boldsymbol{Q}_{i_{nn}}}{2} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We consider the following simple example:

$$\min_{x} \quad 3x_2^2 + 2x_4^2 - 4x_5^2 + x_1x_3 + 2x_1 - 6x_4 + 10$$

Representing with matrices, the above problem can be written as

$$\min_x \begin{pmatrix} 10 & 1 & 0 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix} \bullet \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 & x_1x_5 \\ x_2 & x_2x_1 & x_2^2 & x_2x_3 & x_2x_4 & x_2x_5 \\ x_3 & x_3x_1 & x_3x_2 & x_3^2 & x_3x_4 & x_3x_5 \\ x_4 & x_4x_1 & x_4x_2 & x_4x_3 & x_4^2 & x_4x_5 \\ x_5 & x_5x_1 & x_5x_2 & x_5x_3 & x_5x_4 & x_5^2 \end{pmatrix}$$

$$\Longleftrightarrow \quad \min_x \quad \bar{\boldsymbol{Q}}_0 \bullet \bar{\boldsymbol{X}}.$$

The coefficient matrix $\bar{\boldsymbol{Q}}_0$ has the nonzero diagonal elements in $(3,3), (5,5)$, and $(6,6)$. To apply Theorem 3, we rearrange the diagonal values 3, 2, and -4 as off-diagonal elements with the values of 1.5, 2, and -2, respectively as follows:

$$\bar{\boldsymbol{Q}}_0^* = \begin{pmatrix} 10 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{1.5} & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & \mathbf{-2} \\ 0 & 0 & \mathbf{1.5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{X}}^* \in \mathbb{S}^9 .$$

Note that if there exist $k(< n)$ number of nonzero diagonal elements, then each rearranged element in the off-diagonal does not have to be in line.

The new form of the simple example can be relaxed with the several linear constraints as follows:

$$\begin{aligned} \min_{x} \quad & \bar{\boldsymbol{Q}}_0^* \bullet \bar{\boldsymbol{X}}^* \\ \text{subject to} \quad & \bar{\boldsymbol{X}}^* \succeq 0 \ , \\ & \bar{X}^*_{3,3} = \bar{X}^*_{3,7} \ , \quad \bar{X}^*_{5,5} = \bar{X}^*_{5,8} \ , \quad \bar{X}^*_{6,6} = \bar{X}^*_{6,9} \ . \end{aligned}$$

Lastly, we propose new SOCP and new LP relaxations.

$$\left.\begin{aligned} \zeta^*_{SOCP_{new}} := \min \quad & \bar{\boldsymbol{Q}}_0^* \bullet \bar{\boldsymbol{X}}^* \\ \text{subject to} \quad & \bar{\boldsymbol{Q}}_i^* \bullet \bar{\boldsymbol{X}}^* \leq 0 \ (1 \leq i \leq m), \ \bar{\boldsymbol{H}}_0^* \bullet \bar{\boldsymbol{X}} = 1, \\ & \left\| \begin{pmatrix} \bar{X}^*_{ii} - \bar{X}^*_{jj} \\ 2\bar{X}^*_{ij} \end{pmatrix} \right\| \leq \bar{X}^*_{ii} + \bar{X}^*_{jj}, \\ & (1 \leq i < j \leq 1+n+k), \\ & \bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,\ 1+n+p} \ , \\ & p = 1, \ldots, k, \quad j = \mathcal{C}(p), \\ & where \ \bar{X}^*_{ij} = \bar{X}^*_{ji} \ for \ \forall i \ and \ \forall j \ . \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} \zeta^*_{LP_{new}} := \min \quad & \bar{\boldsymbol{Q}}_0^* \bullet \bar{\boldsymbol{X}}^* \\ \text{subject to} \quad & \bar{\boldsymbol{Q}}_i^* \bullet \bar{\boldsymbol{X}}^* \leq 0 \ (1 \leq i \leq m), \ \bar{\boldsymbol{H}}_0^* \bullet \bar{\boldsymbol{X}}^* = 1, \\ & \bar{X}^*_{ii} \geq 0 \ (1 \leq i \leq 1+n+k), \\ & \bar{X}^*_{ii} + \bar{X}^*_{jj} - 2|\bar{X}^*_{ij}| \geq 0 \\ & (1 \leq i < j \leq 1+n+k), \\ & \bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,\ 1+n+p} \ , \\ & p = 1, \ldots, k, \quad j = \mathcal{C}(p), \\ & where \ \bar{X}^*_{ij} = \bar{X}^*_{ji} \ for \ \forall i \ and \ \forall j \ . \end{aligned}\right\} \tag{15}$$

The formulations (13), (14), and (15) satisfy the DSOS, SDSOS, SDP hierarchies [1] and the assumption of Theorem 3 proposed in Section 3.2. Since nonzero diagonal elements are relocated by the proposed method and the zero diagonal is obtained, Theorem 3 and Corollary 3.1 can be applied to the new SDP, new SOCP and new LP relaxations. Moreover, Definition 2 (SOC) in Section 2.2 can be used to handle second order cone constraints in (14).

From the perspective of the number of constraints, the new SOCP form yields $\frac{(n+k)(n+1+k)}{2}$ and the new LP relaxation yields $(n+1+k)^2$ number of constraints by replacing $\bar{\boldsymbol{X}}^* \succeq 0$ into SOC and linear inequality constraints, respectively. Although the number of constraints in the original problem is smaller, exploiting matrix sparsity can improve the efficiency of the relaxations. But, we note that this approach may be weaker in getting a tight bound, especially for large-size problems. For small-size problems, we expect the difference in the bound quality is not large. We experiment on the proposed method by computing the optimal values in the three relaxations in Section 5, and formulate the three relaxations as the dual problems for Sedumi [9].

4.2 Analysis of the proposed method

In this section, we prove the equivalence of the optimal values between the SDP (3) and the new SDP relaxations (13) obtained by the proposed method. In addition, based on this result, we further extend our result to the new SOCP (14) and the new LP relaxations (15) under certain assumptions.

Theorem 4.1. *Assume that there exist at least one nonzero element in diagonal of* $\boldsymbol{Q}_i'$*s* (2)*. Let* ζ^*_{SDP} *and* $\zeta^*_{SDP_{new}}$ *be the optimal values of SDP* (3) *and new SDP* (13) *relaxations, respectively. Then,* $\zeta^*_{SDP} = \zeta^*_{SDP_{new}}$ *.*

Proof. First, we show that $\zeta^*_{SDP_{new}} \leq \zeta^*_{SDP}$. Let $\bar{\boldsymbol{X}}^*$ be a feasible solution of the new SDP relaxation (13). The constraint $\bar{X}^*_{j+1,j+1} = \bar{X}^*_{j+1,\ 1+n+p}$ in (13) indicates that the objective and constraint functions of the new SDP can be written in a smaller form with less variables, which is the same number as in the original SDP. This follows from that the variables $\bar{X}^*_{j+1,\ 1+n+p}$ and $\bar{X}^*_{1+n+p,\ j+1}$ can be replaced with $\bar{X}^*_{j+1,j+1}$. Moreover, the coefficient of $\bar{X}^*_{j+1,j+1}$ in (13) is changed from zero to two times the coefficient elements of $\bar{X}^*_{j+1,\ 1+n+p}$ since $\bar{Q}^*_{j+1,\ 1+n+p}$ and $\bar{Q}^*_{1+n+p,\ j+1}$ are the same from the symmetry of $\bar{\boldsymbol{Q}}^*_i$'s. The converted problem in this way satisfies constraints in the original SDP relaxation (3). The objective function can be converted, similarly. Thus, we see that the optimal values of the new SDP and the original SDP relaxations are identical.

Next, we have to check whether the positive semidefiniteness of the vairable matrix still holds. We know that the coefficient matrix $\bar{\boldsymbol{Q}}^*_i$ $(i = 1, \ldots, m)$ has no off-diagonal nonzero elements if the indexes of rows and columns exceed $n + 1$. Therefore, only $\bar{X}^*_{j+1,j+1}$ ($j = \mathcal{C}(p)$, $p = 1, \ldots,$ length$(\mathcal{C})$) can remain since they have nonzero coefficients by the substitution. From the positive semidefiniteness of $\bar{\boldsymbol{X}}^*$, a submatrix of $\bar{\boldsymbol{X}}^*$ is also positive semidefinite. Therefore, the new square matrix $\bar{\boldsymbol{X}}$ whose size is smaller than $(n+1, n+1)$ of $\bar{\boldsymbol{X}}^*$ is also positive semidefinite. Therefore, we have found a feasible solution $\bar{\boldsymbol{X}}$ of the original SDP (3). Since $\bar{\boldsymbol{X}}$ does not change the value of the objective and the constraint functions, the set of optimal values in the new SDP is in the set of SDP. Therefore, $\zeta^*_{SDP_{new}}$ is less than or equal to ζ^*_{SDP} .

Conversely, we show that $\zeta^*_{SDP_{new}} \geq \zeta^*_{SDP}$. Let $\bar{\boldsymbol{X}}$ be a feasible solution of the SDP relaxation (3). Consider a new vector $\boldsymbol{Y}$ for the diagonal variables of $\bar{\boldsymbol{X}}$ whose nonzero corresponding diagonal coefficient elements are in the diagonal of $\bar{\boldsymbol{Q}}_i (i = 1, \ldots, m)$ except element located in $(1, 1)$. Let $\mathcal{C}$ be a vector containing the location of nonzero diagonal elements of $\boldsymbol{Q}_i's$ of the QCQP (2). More precisely, $\mathcal{C}_0$ and $\mathcal{C}_i's$ are the vectors containing the location of nonzero diagonal elements in $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_i's$, respectively. Moreover, the size of $\mathcal{C}_0$ and $\mathcal{C}_i's$ are the same as that of $\mathcal{C}$ where $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$. However, if some of the diagonal elements of $\boldsymbol{Q}_i's$ are zeros, the corresponding elements in $\mathcal{C}_i's$ are also zeros. For instance, if elements located in $(2, 2)$ and $(4, 4)$ are zeros in $\boldsymbol{Q}_i$ for some i, then $\mathcal{C}_i(2, 2)$ and $\mathcal{C}_i(4, 4)$ are zeros. Let Y_l be an element in a vector $\boldsymbol{Y}$ where $1 \leq l \leq \text{length}(\mathcal{C})$. Let us define length($\mathcal{C}$) as L. Y_l is nonnegative since it denotes diagonal variable of $\bar{\boldsymbol{X}}$ where $\bar{\boldsymbol{X}}$ is a positive semidefinite matrix. If there exists at least one nonzero diagonal element in the coefficient matrices $\boldsymbol{Q}_i's$, we can say that $Y_1 = \boldsymbol{X}_{\mathcal{C}(1),\mathcal{C}(1)}$, where $\boldsymbol{X}$ is a submatrix of $\bar{\boldsymbol{X}} = \begin{pmatrix} x_{00} & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix}$.

Then the problem of the SDP relaxation (3) can be written as

$$
\begin{aligned}
\min_{x} \quad & \boldsymbol{r}_0 + \boldsymbol{q}_0^T (X_{11}, \ldots, X_{1n})^T + \sum_{\substack{u=2\\u\neq v}}^{n+1} \sum_{v=2}^{n+1} Q_{0_{uv}} \bar{X}_{uv} \\
& + \sum_{l=1}^{L} Q_{0_{\mathcal{C}_0(l),\mathcal{C}_0(l)}} Y_l \\
\text{subject to} \quad & \boldsymbol{r}_i + \boldsymbol{q}_i^T (X_{11}, \ldots, X_{1n})^T + \sum_{\substack{u=2\\u\neq v}}^{n+1} \sum_{v=2}^{n+1} Q_{i_{uv}} \bar{X}_{uv} \\
& + \sum_{l=1}^{L} Q_{i_{\mathcal{C}_i(l),\mathcal{C}_i(l)}} Y_l \leq 0 \qquad (i = 1, ..., m), \\
& \bar{\boldsymbol{H}}_0 \bullet \bar{\boldsymbol{X}} = 1, \\
& \bar{\boldsymbol{X}} \succeq \boldsymbol{O}, \\
& Y_l = X_{\mathcal{C}(l),\mathcal{C}(l)} \;\; (l = 1, \ldots L),
\end{aligned}
$$

where $Q_{0_{0,0}}$ and $Q_{i_{0,0}}$ are zeros. The new vector $\boldsymbol{Y}$ and the variables $\bar{X}_{uv}(u \neq v)$ satisfy the constraints of linear constraints in the new SDP relaxation (13) if we identify Y_l and $X_{\mathcal{C}(l),\mathcal{C}(l)}$ with $\bar{X}^*_{j+1,1+n+p}$ and $\bar{X}^*_{j+1,j+1}$, respectively.

Now, we have to show that the new matrix $\bar{\boldsymbol{X}}^*$ is positive semidefinite. Since $\bar{\boldsymbol{X}}$ is positive semidefinite, we can construct a matrix $\bar{\boldsymbol{X}}^*$ such as

$$
\bar{\boldsymbol{X}}^* = \begin{pmatrix} \bar{\boldsymbol{X}} & diag(\boldsymbol{Y})^T \\ diag(\boldsymbol{Y}) & \beta \boldsymbol{I} \end{pmatrix},
$$

where

$$
\beta = \max \left\{ \frac{Y_1^2}{X_{\mathcal{C}(1),\mathcal{C}(1)}}, \frac{Y_2^2}{X_{\mathcal{C}(2),\mathcal{C}(2)}}, \ldots, \frac{Y_L^2}{X_{\mathcal{C}(L),\mathcal{C}(L)}} \right\},
$$

and the size of diag($\boldsymbol{Y}$) is $L \times (n+1)$. Note that the first row of diag($\boldsymbol{Y}$) is zero and the variable Y_l starts from the second row because vector $\boldsymbol{Y}$ is from $\boldsymbol{X}$ where $\boldsymbol{X}$ is a submatrix of $\bar{\boldsymbol{X}}$. Furthermore, diag($\boldsymbol{Y}$) does not need to be written consecutively on diagonal lines since nonzero $X_{\mathcal{C}(l),\mathcal{C}(l)}$, corresponding to the nonzero coefficient diagonal elements in $\boldsymbol{Q}_i's$, may not exist in a row. The β shown above satisfies the positive semidefiniteness of $\bar{\boldsymbol{X}}^*$ because all 2×2 submatrix of $\bar{\boldsymbol{X}}^*$ is greater than or equal to zero. Accordingly, we can establish a feasible solution $\bar{\boldsymbol{X}}^*$ of the new SDP relaxation (13) whose objective value is the same as the original SDP relaxation (3). Since $\bar{\boldsymbol{X}}^*$ is positive semidefinite and it satisfies the objective and the constraint functions in (13), $\zeta^*_{SDP_{new}} \geq \zeta^*_{SDP}$ follows.

□

Corollary 4.1. *Assume that there exist at least one nonzero element in some coefficient matrices of the original QCQP* (2)*. Then,* $\zeta^*_{SDP} = \zeta^*_{SDP_{new}} = \zeta^*_{SOCP_{new}} = \zeta^*_{LP_{new}} = \mu^*_{SDP_{new}} = \mu^*_{SOCP_{new}} = \mu^*_{LP_{new}}$ *under theorem 4.1 .*

Proof. We have shown in Theorem 4.1 that if there exist some nonzero diagonal elements in $\boldsymbol{Q}_i's$ (2), we can move them to off-diagonal to set the new SDP relaxation using proposed method, and can get the same optimal value of the SDP relaxation. Moreover, by Corollary 3.1, we know that the primal and the dual optimal values of the new SDP, new SOCP, and new LP relaxations are the same because all the diagonal elements in the coefficient matrices can be made to zeros. Therefore, we have that $\zeta^*_{SDP} = \zeta^*_{SDP_{new}} = \zeta^*_{SOCP_{new}} = \zeta^*_{LP_{new}} = \mu^*_{SDP_{new}} = \mu^*_{SOCP_{new}} = \mu^*_{LP_{new}}$.

□

5 Numerical Results

We present computational results on the SDP relaxation (3), and compare it with the results of the new SDP (13), SOCP (14), and LP (15) relaxations using the proposed method in Section 4. In the SDP relaxation, we deal with matrices $\boldsymbol{Q}_i$ $(i = 1, \ldots, m)$ which are not hollow matrices.

The set of test problems in our numerical experiments consist of small problems where the number of values are less than or equal to 30. To attain numerical feasibility, we set the number of constraints m to approximately $(n+1) \times (n+1)$.

Notation

n	The number of variables
m	The number of quadratic inequality constraints
nz	The total number of nonzero diagonal elements in $\boldsymbol{Q}_i's$ of a QCQP
sp.	The density of nonzeros in a matrix
cpu	The CPU time in seconds
it.	The number of iterations that the corresponding relaxation takes
SDP	The SDP relaxation
SDP^*	The new SDP relaxation
$SOCP^*$	The new SOCP relaxation
LP^*	The new LP relaxation
ζ^*	The optimal value of a relaxation
ζ^*_{SDP}	The optimal value of SDP relaxation
$\zeta^*_{SDP_{new}}$	The optimal value of new SDP relaxation
$\zeta^*_{SOCP_{new}}$	The optimal value of new SOCP relaxation
$\zeta^*_{LP_{new}}$	The optimal value of new LP relaxation

We compared the optimal values, computational time, and number of iterations obtained from executing the original SDP (3), new SDP (13), new SOCP (14), and new LP (15) relaxations by varying the number of variables, nonzero diagonal elements, constraints, and the degree of matrix sparsity. To control the sparsity of the symmetric matrices $\boldsymbol{Q}_i$ $(i = 1, \ldots, m)$, we used MATLAB random number generator. All the computation was implemented using a MATLAB toolbox, SeDuMi Version 1.03 [9] on Window 7 with Intel Core i5-2467M CPU, 1.60 GHz, memory 8GB.

First, we tested whether the result of optimal values are equivalent between the SDP and the new SDP relaxations. Next, we compared computational time and the iteration numbers of each relaxations to see whether the proposed method is efficient for randomly generated problems. Note that each problems are derived from nonconvex QCQPs where the symmetric matrices $\boldsymbol{Q}_i's$ in (2) has at least one negative eigenvalue.

Table 1: QCQPs with nz=3, 50% sparsity and varying n

		SDP		*SDP**		*SOCP**		*LP**	
n	ζ^*	cpu	it.	cpu	it.	cpu	it.	cpu	it.
5	8.68E+01	1.0	18	0.3	19	0.6	15	0.2	11
9	1.94E+01	1.2	19	0.4	21	0.9	17	0.3	11
14	-1.05E+02	0.9	8	0.4	8	1.0	12	0.3	4
19	7.04E+01	2.9	23	3.2	23	4.5	19	2.7	13
23	-1.49E+02	3.1	9	3.3	10	5.9	12	3.0	6
27	7.87E+01	12.5	23	17.4	25	20.1	17	19.7	14

First of all, we generated the coefficient matrices $\boldsymbol{Q}_i$ $(i = 1, \ldots, m)$ with 50% sparsity, and fixed the total number of nonzero diagonal elements (nz) as three. Note that the number of coefficients m varied in each problem to prevent primal or dual infeasibility. Then, we changed n to see how the speed of computation differs. Table 1 above shows that as the number of variables increases, the computational time also increases to get feasible solutions. In addition, we observed that the optimal values of the four relaxations have the same value, $\zeta^*_{SDP} = \zeta^*_{SDP_{new}} = \zeta^*_{SOCP_{new}} = \zeta^*_{LP_{new}}$, as shown in Theorem 4.1 and Corollary 4.1. Thus, we let ζ^* be the optimal value of the four relaxations. Although the new SDP and the new SOCP relaxations spent more CPU time and had larger number of iterations for some n such as 23, 27, the new LP relaxation can be used for those cases instead as the optimal value is equal to the original SDP relaxation along with less computational time and number of iterations.

Secondly, we generated 20% sparse coefficient matrices $\boldsymbol{Q}_i$ $(i = 1, \ldots, m)$, and fixed the total number of nonzero diagonal elements (nz) as seven. For fair comparison, we also fixed n as fifteen and varied the number of coefficients m greater than or equal to 230 to prevent primal or dual infeasibility. The results are as follows:

Table 2: QCQPs with n=15, nz=7, 20% sparsity and varying m

		SDP		*SDP**		*SOCP**		*LP**	
m	ζ^*	cpu	it.	cpu	it.	cpu	it.	cpu	it.
230	3.54E-01	1.3	23	1.6	23	2.5	19	1.1	12
260	1.79	0.9	9	0.6	9	1.4	8	0.8	5
360	-1.58E-01	0.9	8	0.6	8	0.7	7	0.2	4
460	6.29E+01	0.9	7	0.6	7	0.8	9	0.3	4
550	3.03E+01	1.6	6	0.7	7	2.0	13	0.3	4
800	4.27	1.8	6	0.8	7	0.8	8	0.3	4

In Table 2, we assumed that larger m may give a bounded region easily than having a small number of constraints. We also detected the new LP relaxation can be a substitution for the SDP and the SOCP relaxations since it yielded less CPU time and the number of iterations with the same optimal value. However, the new SDP and the new SOCP relaxations are not much different from the original SDP relaxation.

Table 3: QCQPs with n=20, m=450, nz=5 and varying sparsity

		SDP		*SDP**		*SOCP**		*LP**	
$sp.$(%)	ζ^*	cpu	it.	cpu	it.	cpu	it.	cpu	it.
10	2.74E+01	2.3	25	3.5	25	2.7	18	1.5	14
20	2.55E+02	1.5	8	1.5	9	1.9	10	0.6	4
40	1.34	2.4	14	2.2	13	2.3	10	1.0	5
50	-5.12E-01	2.6	17	2.8	17	3.2	13	1.5	6
70	8.71E-01	2.3	16	2.2	16	4.7	14	1.8	6
85	-1.63	2.7	15	2.9	15	4.4	12	2.4	7
100	4.47E-01	2.1	10	1.7	9	3.6	10	2.0	5

Third, Table 3 shows the results from varying sparsity of $\boldsymbol{Q}_i$ $(i = 1, \ldots, m)$ from 10% to 100%, fully dense matrix $\boldsymbol{Q}_i$. We fixed the number of variables (n), number of constraints (m), and number of nonzero diagonal elements (nz) in the coefficient matrices of (2). Since 100% sparsity means fully dense, CPU time in the new SOCP relaxation took more time than the SDP relaxation in the process of applying the proposed method. This is because Second Order Cone (SOC) constraints in (14) are derived from nonzero elements in matrices, and so the dense matrices yield more SOC constraints. The new LP relaxation (15) also has more constraints than the SDP (3) and the new SDP (13) relaxations, however, it can be an efficient method among other relaxations since it made the better performance with the same optimal value.

Lastly, we changed the number of nonzero elements (nz) in the diagonal of $\boldsymbol{Q}_i's$ with fixed n as 25, m as 650, and matrix sparsity as 10%. Note that we only considered small problems where n, the number of variables, is less than 30 at the beginning of Section 5. The reason of this restriction is to avoid making large sized matrices which require more computational time. Also, the proposed method makes new larger matrices $\bar{\boldsymbol{Q}}_i^*$ by counting the number of nonzero diagonal elements (nz) in the coefficient matrices of the original QCQP.

Table 4: QCQPs with n=25, m=650, 10% sparsity and varying nz

		SDP		*SDP**		*SOCP**		*LP**	
nz	ζ^*	cpu	it.	cpu	it.	cpu	it.	cpu	it.
1	5.41E+01	3.6	9	2.7	10	3.8	11	2.1	6
3	4.63E-01	2.3	9	1.6	9	2.5	9	1.6	7
5	1.56E+01	2.4	9	4.5	11	5.2	12	2.1	5
8	1.20E+01	3.5	20	8.3	19	5.2	16	2.6	11
10	-6.06	4.8	17	8.4	17	7.1	14	2.5	10
13	5.91	2.3	8	6.6	8	17.2	9	11.7	6
18	1.80E-01	2.1	7	10.5	7	16.5	6	10.5	5
25	3.42	1.3	10	31.4	10	14.3	13	0.4	9

Table 4 shows the results of four relaxations from varying the number of nonzero diagonal elements (nz) in $\boldsymbol{Q}_i's$. For example, if n is 25 and nz is 10, size of the new matrix $\bar{\boldsymbol{Q}}_i^*(i = 1, \ldots, m)$ becomes 36×36 where size of the original matrix $\bar{\boldsymbol{Q}}_i$ is 26×26. For small number of nz such as 1, 3, 5, 8 and 10, the new LP relaxation shown in the right side of the Table 4 has less CPU time and iteration numbers than other relaxations to predict the optimal value. On the contrary, QCQPs with the bigger nz such as 13, 18 or more, took a lot of CPU time to get the optimal value. One good thing to implement the new LP relaxation for large nz is that the number of iterations is lesser than the SDP relaxation. This indicates that as the size of nz became larger and approached to n, the original SDP relaxation required less CPU time than implementing the proposed method to solve the nonconvex problem. Nonetheless, if we have small nz in QCQPs, we can expect much less computational effort and number of iterations with the same quality of the optimal value from the new LP relaxation instead of implementing the SDP, new SDP, and new SOCP relaxations. Moreover, even for large nz, less number of iterations was required in the new LP relaxation than the SDP relaxation of nonconvex QCQPs.

6 Conclusion

We have proposed a method to solve nonconvex QCQP efficiently. Based on the result on nonconvex QCQPs with hollow coefficient matrices, our method converts QCQP with nonzero diagonal coefficient matrices to QCQP with hollow matrices. By the conversion, we can utilize more efficient convex relaxation methods such as SOCP and LP relaxation which are shown to provide the same objective values as the SDP relaxation. Moreover, we have proved that the optimal value of the SDP relaxation of the original QCQP is equivalent to that of the new SDP relaxation, consquently, those of the SOCP and LP relaxation.

We have also tested to exploit the sparsity of the converted QCQP to solve them with high efficiency. With numerical experiements, we have demonstrated that the proposed method can perform efficiently compared to the SDP relaxation of the original problem.

For further research direction, we can apply our method to large-size QCQP and investigate the quality of the optimal values. In particular, it will be interesting to apply our method to the framework of the hierarchy of SDP relaxation proposed by [8].

References

[1] A. A. Ahmadi and A. Majumdar. "DSOS and SDSOS optimization: LP and SOCP-based alternatives to sum of squares optimization." In *2014 48th annual conference on information sciences and systems (CISS)*, pp. 1-5. IEEE, 2014.

[2] A. A. Ahmadi, S. Dash and G. Hall. "Optimization over structured subsets of positive semidefinite matrices via column generation." *Discrete Optimization* 24, pp. 129-151, 2017.

[3] ApS, Mosek. "The MOSEK optimization toolbox for MATLAB manual." *Version 7.1 (Revision 28)* 5, 2015.

[4] C. A. Haverly. "Studies of the behavior of recursion for the pooling problem." *Acm sigmap bulletin* 25, pp. 19-28, 1978.

[5] E. L. Lawler. "The quadratic assignment problem." *Management science* 9, no. 4, pp. 586-599, 1963.

[6] G. Barker and D. Carlson. "Cones of diagonally dominant matrices." *Pacific Journal of Mathematics* 57, no. 1, pp. 15-32, 1975.

[7] I. M. Bomze, M. Budinich, P. M. Pardalos, and M. Pelillo. "The maximum clique problem." In *Handbook of combinatorial optimization*, pp. 1-74, Springer, Boston, MA, 1999.

[8] J. B. Lasserre. "Global optimization with polynomials and the problem of moments." *SIAM Journal on optimization* 11, no. 3, pp. 796-817, 2001.

[9] J. F. Sturm. "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones." *Optimization methods and software* 11, no. 1-4, pp. 625-653, 1999.

[10] K. C. Toh, M. J. Todd, and R. H. Tütüncü. "SDPT3—a MATLAB software package for semidefinite programming, version 1.3." *Optimization methods and software* 11, no. 1-4, pp. 545-581, 1999.

[11] M. Kimizuka, S. Kim, and M. Yamashita. "Solving Pooling Problems by LP and SOCP Relaxations and Rescheduling Methods." *arXiv preprint arXiv:1804.02857*, 2018.

[12] M. Muramatsu and T. Suzuki. "A new second-order cone programming relaxation for max-cut problems." *Journal of the Operations Research Society of Japan* 46, no. 2, pp. 164-177, 2003.

[13] M. X. Goemans and D. P. Williamson. "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming." *Journal of the ACM (JACM)* 42, no. 6, pp. 1115-1145, 1995.

[14] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge university press, 2004.

[15] S. Kim and M. Kojima. "Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations." *Computational Optimization and Applications* 26, no. 2, pp.143-154, 2003.

[16] S. Kim and M. Kojima. "Exploiting sparsity in SDP relaxation of polynomial optimization problems." *In Handbook on Semidefinite, Conic and Polynomial Optimization*, pp. 499-531. Springer, Boston, MA, 2012.

[17] S. Kim and M. Kojima. "Second order cone programming relaxation of non-convex quadratic optimization problems." *Optimization methods and software* 15, no. 3-4, pp. 201-224, 2001.

[18] S. Zhang. "Quadratic maximization and semidefinite relaxation." *Mathematical Programming* 87, no. 3, pp. 453-465, 2000.

[19] T. Nishi. "A semidefinite programming relaxation approach for the pooling problem." PhD diss., Master's thesis, Department of Applied Mathematics and Physics, Kyoto University, 2010.

[20] Y. Wang, C. Zhang, and Z. Liu. "A matrix approach to graph maximum stable set and coloring problems with application to multi-agent systems." Automatica 48, no. 7, pp. 1227-1236, 2012.

국 문 초 록

수 학 과

이 목 화

2차 제약 조건을 가진 2차 계획법(Quadratically Constrained Quadratic Program, QCQP)의 전역 최소치를 찾는 것은 이진 최적화 문제(Binary Quadratic Optimization Problems), 최대 절단 문제(Max-cut Problems) 및 네트워크 문제(Network Problems)의 응용에서 매우 중요한 부분을 차지한다. 지금까지 다양한 각도에서 2차 계획법에 대한 최적의 해를 찾을 수 있는 방법이 고안되어 왔지만, 위와 같은 NP-hard 문제는 복잡성이라는 성질 때문에 이를 다항 시간 안에 해결하기 위한 일반적인 알고리즘이 아직까지 제시되지 않았다. 따라서, 주어진 문제에 대한 정확한 해를 찾는 대신 이완법(Relaxation Method)이 제안되어왔고, 이는 많은 연구자들에 의해 이론적 효과가 증명되었다. 가장 주목할만한 이완 방법은 준정부호(Semidefinite Programming, SDP) 이완법, SOCP(Second Order Cone Programming, SOCP) 이완법 및 선형계획(Linear Programming, LP) 이완법이다. 이 논문에서는 QCQP를 풀기 위해 위의 세 가지 이완법에 대한 새로운 접근 방법을 제시하였고, 행렬의 희소성을 사용하여 계산 효율성을 향상시켰다. 특히, 우리는 Kimizuka, Kim, Yamashita이 2018년 연구한 SDP, SOCP 및 LP 이완법의 관계를 사용해 특정 가정 하에서 새롭게 제시된 SDP, SOCP 및 LP 이완법의 최적 값의 등가성을 확인했다.